GEODESIC FLOWS ARE BERNOULLIAN

BY

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ABSTRACT

A geometric method is developed for proving that transformations are isomorphic to Bernoulli shifts. The method is applied to the geodesic flows on surfaces of negative curvature and it is shown that they are isomorphic to Bernoulli flows.

The existence of Bernoulli flows, i.e., measure-preserving flows ϕ_t such that for each fixed t we have ϕ_t isomorphic to a Bernoulli shift, was recently established [8] by constructing a special flow over a B-shift. The problem immediately arose concerning whether or not any of the classical flows of ergodic theory are Bernoulli, and it is the purpose of this investigation to give an affirmative answer. We develop a geometric method for proving that a mapping is isomorphic to a B-shift and apply it to show that the geodesic flow on a surface of negative curvature is a Bernoulli flow.† This means that a mechanical system which seems to be quite regular is actually the same, from the point of view of ergodic theory, as the most random type of process possible. Because of the isomorphism theorem for Bernoulli flows [11], our theorem tells us exactly (up to isomorphism) what the geodesic flow on a surface of negative curvature is. It is (after a normalization of the time parameter) isomorphic to the simple flow described in [8].

An attempt has been made to make the paper accessible to a wider circle of readers and so we have not striven for greatest generality. With some modifications the techniques used extend to show that a wide class of Anosov diffeomorphisms and flows are isomorphic to Bernoulli systems; we hope to return to these matters in the future.

† This result was announced in [9] but the argument envisioned then was incomplete.

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The necessary preliminaries from the theory of isomorphisms in ergodic theory are given in §1, which is self-contained but for the proofs of Theorems A and B. To clarify the ideas we reprove in §2 the known result that ergodic automorphisms of the 2-torus are isomorphic to B-shifts. In §3 we give the main result that the geodesic flow on a compact 2-dimensional manifold of negative curvature is a Bernoulli flow. Some acquaintance with the properties of this flow is assumed in the discussion.

1. Approximate independence and Bernoulli shifts

In this section $(X,\mathscr{B},\mu)$ is a fixed measure space, $\mu(X)=1$, and $\phi: X \to X$ is an invertible measure-preserving transformation. Partitions of X will be denoted by lower case Greek letters, $\alpha,\beta,\gamma,\cdots$ $\alpha=\{A_1,A_2,\cdots A_a\}$, $\beta=\{B_1,B_2,\cdots B_b\},\cdots$. If every atom of α is a union of atoms of β then we write $\alpha\subseteq\beta$, and $\alpha\vee\beta$ denotes the least common refinement of α and β, i.e. the partition into sets $A_i\cap B_j$. In case $\{\alpha_i\}$ is an infinite sequence of partitions, $\bigvee\alpha_i$ denotes the smallest σ-algebra with respect to which all the α_i are measurable. A partition α is said to be a *generator* (this depends on ϕ) if $\bigvee_{-\infty}^{\infty}\phi^n\alpha=\mathscr{B}$, where $\phi^n\alpha=\{\phi^nA_1,\cdots\phi^nA_a\}$. Partitions will be thought of as coming equipped with an ordering of their atoms and this is inherited in a lexicographic fashion by $\alpha\vee\beta$.

A partition α is said to be *independent* for ϕ if for all choices of i_j, $-n\leqq j\leqq n$, all n

$$\mu\left(\bigcap_{j=-n}^{n}\phi^{-j}A_{i_j}\right)=\prod_{-n}^{n}\mu(\phi^{-j}A_{i_j}). \tag{1}$$

If ϕ has an independent generator then ϕ is said to be a Bernoulli shift, or B-shift. Before formulating some weaker notions of independence let us introduce a term for a situation that will arise frequently. If a property P holds for all atoms of a partition α with the possible exception of a set of atoms whose union has measure less than ε, then we shall say that P holds for *ε-almost every* atom of α, or that for *ε-a.e.* atom A of α, P holds. A partition α is a *K-partition* if for any $B\in\bigvee_{-\infty}^{\infty}\phi^{-n}\alpha$, given $\varepsilon>0$ there is an $N_0=N(\varepsilon,B)$ such that for all $N'\geqq N\geqq N_0$ and ε-a.e. atom A of $\bigvee_N^{N'}\phi^k\alpha$ we have

$$\left|\mu(A\cap B)/\mu(A)-\mu(B)\right|<\varepsilon. \tag{2}$$

Clearly B can be replaced by any finite collection of sets. There is a theorem due to Pinsker, Rohlin and Sinai ([12], [14]) that says in part, that if ϕ has a generator

α which is a K-partition then *every* partition is a K-partition. In that case we say that ϕ is a K-automorphism (K-for Kolmogorov, who introduced the notion). Not every K-automorphism is a B-shift, (see [10]), but if we strengthen somewhat the approximate independence we will get a property which does imply that ϕ is a B-shift. First we need to recall some distance functions on the space of partitions. The usual metric is

$$\mathrm{d}(\alpha,\beta) = \sum_i \mu(A_i \Delta B_i) \tag{3}$$

where $A \Delta B$ is the symmetric difference, $(A \cup B) \backslash A \cap B$. Given two sequences of partitions $\{\alpha_i\}_1^n$, $\{\beta_i\}_1^n$ possibly on different spaces, we could compare their distributions, and we will write

$$\{\alpha_i\}_1^n \sim \{\beta_i\}_1^n \tag{4}$$

if for all k_i, $1 \leqq i \leqq n$

$$\mu\left(\bigcap_1^n A_{k_i}^{(i)}\right) = \nu\left(\bigcap_1^n B_{k_i}^{(i)}\right) \tag{5}$$

where μ, ν are the measures on the spaces X, Y, $\alpha_i = \{A_1^{(i)}, \cdots A_{a_i}^{(i)}\}$ are partitions of X, and $\beta_i = \{B_1^{(i)}, \cdots B_{b_i}^{(i)}\}$ are partitions of Y. We weaken this notion by writing

$$\bar{d}(\{\alpha_i\}_1^n, \{\beta_i\}_1^n) \leqq \varepsilon \tag{6}$$

if there are $\bar{\alpha}_i$, $\bar{\beta}_i$ partitions on the same space such that $\{\alpha_i\}_1^n \sim \{\bar{\alpha}_i\}_1^n$, $\{\beta_i\}_1^n \sim \{\bar{\beta}_i\}_1^n$ and

$$1/n\left(\sum_1^n \mathrm{d}(\bar{\alpha}_i, \bar{\beta}_i)\right) \leqq \varepsilon. \tag{7}$$

If E is any subset of X then α/E means the partition of E into sets of the form $A \cap E, A \in \alpha$, where the measure on E has been renormalized so that E has measure one. A partition α is said to be *very weakly Bernoullian* (VWB) if for every $\varepsilon > 0$ there is an $N_0 = N_0(\varepsilon)$ such that for all $N' \geqq N \geqq N_0$ and all $n \geqq 0$, ε-a.e. atom A of $\bigvee_N^{N'} \phi^k \alpha$ satisfies

$$\bar{\mathrm{d}}(\{\phi^{-i}\alpha\}_1^n, \{\phi^{-i}\alpha/A\}_1^n) \leqq \varepsilon. \tag{8}$$

We can now formulate the main tools at our disposal for showing that transformations are B-shifts. They are the following two theorems, to be found in [8] and [7] respectively.

THEOREM A. *If α is VWB then $(X, \bigvee_{-\infty}^{\infty} \phi^n \alpha, \mu, \phi)$ is a B-shift.*

THEOREM B. *If $\mathscr{A}_1 \subseteq \mathscr{A}_2 \subseteq \cdots$ are an increasing sequence of ϕ-invariant* *σ-algebras,* $\bigvee_1^\infty \mathscr{A}_n = \mathscr{B}$ *and for each* $n, (X, \mathscr{A}_n, \mu, \phi)$ *is a B-shift, then* $(X\ \mathscr{B}, \mu, \phi)$ *is a B-shift.*

To apply Theorem A we need a method for showing that partitions are close in the d-metric and we devote the rest of the section to this point. The $\{\alpha_i\}_1^n$-name of x is the sequence $l_i = l_i(x)$ determined by

$$x \in A_{l_i}^{(i)},\ \alpha_i = \{A_1^{(i)},\ A_2^{(i)}, \cdots A_{a_i}^{(i)}\}. \tag{9}$$

Let e be defined on the integers by $e(j) = 1$ for $j \neq 0$ and $e(0) = 0$.

LEMMA 1.1. *Let* $\{\alpha_i\}_1^n$ *be partitions of* X *with name functions* $l_i(x)$, *and* $\{\beta_i\}_1^n$ *be partitions of* Y *with name functions* $m_i(y)$. *If there is a measure-preserving mapping* $\theta: X \to Y$ *such that*

$$\begin{cases} 1/n \sum_1^n e(l_i(x) - m_i(\theta x)) \leqq \varepsilon, & x \in X \setminus E \\ \mu(E) \leqq \varepsilon \end{cases} \tag{10}$$

then

$$d(\{\alpha_i\}_1^n,\ \{\beta_i\}_1^n) \leqq 4\varepsilon. \tag{11}$$

PROOF. Let $\bar{\alpha}_i = \theta\alpha_i$, $\bar{\beta}_i = \beta_i$. Since θ is measure-preserving, $\{\theta\alpha_i\}_1^n \sim \{\alpha_i\}_1^n$; in fact, this is the only measure-preserving property of θ needed here. Note that for two partitions α, β, if $f(x) = 1$ when the α-name disagrees with the β-name and $f(x) = 0$ when the names agree, then

$$d(\alpha, \beta) = 2 \int f(x)\mu(dx). \tag{12}$$

Observe that for $\bar{\alpha}_i$ and $\bar{\beta}_i$,

$$f_i(y) = e(l_i(\theta^{-1}y) - m_i(y)) \tag{13}$$

is 1 if the $\bar{\alpha}_i$-name disagrees with the $\bar{\beta}_i$-name and is zero otherwise. Combining this observation with (12) and (10) we obtain (11). □

A version of Lemma 1 will be needed in which the mapping θ is not quite measure-preserving. We start with a definition. A mapping $\theta: X_1 \to X_2$ will be called *ε-measure-preserving* if there is a set $E_1 \subset X_1$, $\mu_1(E_1) \leqq \varepsilon$ and for all $A \subset X_1 \setminus E_1$

$$|\mu_2(\theta A)/\mu_1(A) - 1| \leqq \varepsilon. \tag{14}$$

The next lemma is proved by a standard argument and the proof is omitted.

LEMMA 1.2. *If* $\theta: X_1 \to X_2$ *is ε-measure-preserving, and* α *is a partition of* X_1, *then there is a mapping* $\bar{\theta}: X_1 \to X_2$ *such that*

$$\{\bar{\theta}\alpha\} \sim \{\alpha\} \tag{15}$$

and

$$\mu_1\{x_1 : \theta x_1 \neq \bar{\theta} x_1\} \leqq 3\varepsilon. \tag{16}$$

Note that, in general, $\bar{\theta}$ cannot be chosen to satisfy (16) and (15) for *all* α simultaneously. Now we strengthen Lemma 1.1.

LEMMA 1.3. *Let $\{\alpha_i\}_1^n$ be partitions of X with name functions $l_i(x)$, and $\{\beta_i\}_1^n$ be partitions of Y with name functions $m_i(y)$. If there is an ε-measure-preserving mapping $\theta: X \to Y$ so that*

$$\begin{cases} 1/n \sum_1^n e(l_i(x) - m_i(\theta x)) \leqq \varepsilon, & x \in X \setminus E \\ \mu(E) \leqq \varepsilon \end{cases} \tag{17}$$

then

$$d(\{\alpha_i\}_1^n, \{\beta_i\}_1^n \leqq 16\varepsilon. \tag{18}$$

PROOF. Apply Lemma 1.2 with α replaced by $\bigvee_1^n \alpha_i$. Notice that (17) and (16) imply that (10) holds for $\bar{\theta}$ with ε replaced by 4ε. As we have remarked, the proof of Lemma 1.1 will apply to $\bar{\theta}$ and thus (18) follows. □

In applying Lemma 1.3, one of the measure spaces will be a subset of a given measure space, and then ε-measure-preserving should be understood with the measures involved renormalized so that they all have total measure one. The knowledgeable reader will no doubt wonder that no mention has yet been made of entropy. All of the systems that will be discussed have finite entropy and thus the results of [11] on isomorphisms of Bernoulli flows together with Theorem 3.1 imply that, for some scaling of the time parameter, any two geodesic flows are isomorphic. Indeed, in [11] it is shown that if $h(\phi_1) = h(\tilde{\phi}_1)$ for two flows and ϕ_1 and $\tilde{\phi}_1$ are both B-shifts then the flows are isomorphic. Since $h(\phi_t) = |t| h(\phi_1)$, entropy provides the right scaling of the time parameter. To be sure, entropy plays a decisive role in the proofs of Theorem A and B, but no knowledge of it is required in the applications, as we shall see below.

2. Automorphisms of the 2-torus are B-shifts

The method that will be used to prove that certain transformations are B-shifts applies to those oft-studied transformations, the automorphisms of the torus.

Since the idea of the proof can be seen most clearly in this simple case we devote this section to a proof of

THEOREM 2.1. *An ergodic automorphism of the 2-torus equipped with Lebesgue measure is a B-shift.*

Henceforth, we shall be dealing only with topological spaces and the σ-algebra of measurable sets will always be taken to be the Borel σ-algebra. In [1] it was shown that these automorphisms are Markov shifts while in [6], Theorem 2.1 was proved for the n-torus. The method of proof given below applies to automorphisms of the n-torus that have no repeated eigenvales of modulus one. For that case, it seems that the harmonic analysis of [6] is the most convenient tool. Before coming to the details, let us sketch the idea of the proof. By Theorems A and B of §1, it suffices to show that a refining sequence of partitions of the torus is VWB. Let α denote a fixed partition and A an atom of $\bigvee_N^{N'}\phi^k\alpha$. By Lemma 1.3, to see that α is VWB we must map A onto the torus by a mapping θ that is nearly measure preserving and that has the additional feature of keeping x and θx close together under the action of ϕ^k, $k \geqq 0$. The construction of θ is done in two steps. First, an auxiliary partition β is chosen which depends on ε. Then the fact that the automorphism is a K-automoıphism is used to give a rough, nearly measure-preserving mapping from A onto atoms of β by mapping $A \cap B$ onto B. The contracting and expanding foliations are used "locally" to give a nearly measure-preserving mapping θ from $A \cap B$ to B such that $\{\phi^k x\}_1^n$ is close to $\{\phi^k \theta x\}_1^n$. This is done by keeping θx on the contracting fiber that contains x. After this sketch we turn to the details.

Let X be the 2-torus and μ the usual Lebesgue measure on X. An algebraic automorphism ϕ of the torus X is defined by an integral unimodular 2×2 matrix Φ. Since Φ is unimodular it preserves μ and we have a m.p.t. (X, μ, ϕ) which we wish to show is a B-shift. The key to the proof is provided by certain invariant fiberings or foliations which we proceed to describe. The ergodicity of ϕ means that no eigenvalue of Φ can be a root of unity, which in this case means that Φ has two eigenvalues λ_1, λ_2 with $|\lambda_1| < 1 < |\lambda_2|$. Consider Φ as a linear transformation of R^2 and then project the families of lines in the plane parallel to the characteristic directions of Φ to get two ϕ-invariant fiberings of the torus, $\mathscr{E}$ and $\mathscr{C}$. Let $\mathscr{E}_x$ denote the line through $x \in X$ that is parallel to the characteristic direction associated with λ_2. Since $|\lambda_2| > 1$, it expands when ϕ is applied to it; that is, if $y_1, y_2 \in \mathscr{E}_x$ are separated by a distance d measured along $\mathscr{E}_x$, ϕy_1 and ϕy_2 which lie on $\mathscr{E}_{\phi x}$ are separated by $|\lambda_2|\cdot d$, along $\mathscr{E}_{\phi x}$. $\mathscr{C}$ corresponds in a similar way to λ_1, and these are just the simplest kind of expanding and contracting foliations for a transformation ϕ. With $\mathscr{E}$ and $\mathscr{C}$ fixed, a set A will be called a *parallelogram* if it is connected, its closure equals the closure of its interior points and for any $x, y \in A$ the connected component of $\mathscr{C}_x \cap A$ that contains x intersects the connected component of $\mathscr{E}_y \cap A$ that contains y, in a unique point $z \in A$.

CONVENTION. If A is a parallelogram then for $x \in A$, $\mathscr{E}_x \cap A$ will denote the connected component of $\mathscr{E}_x \cap A$ that contains x. Similarly for $\mathscr{C}_x \cap A$.

Let $\alpha = \{A_1, \cdots A_a\}$ be a fixed partition of X which we want to show is VWB. The only regularity condition that we shall impose on α is (RA): Each atom is a set with dense interior points and the boundary of A_i consists of a finite number of smooth curves. Suppose that $\varepsilon > 0$ is given and that β is an auxiliary partition which will depend on ε. For the meantime we merely assume that $\beta = \{B_0, B_1, \cdots B_b\}$ where the B_i are parallelograms, for $1 \leqq i \leqq b$, and the measure of B_0 is small. A set $E \subset X$ is said to intersect B_i in an *$\mathscr{E}$-tubular subset* of B_i if $x \in E \cap B_i$ implies that $\mathscr{E}_x \cap B_i \subset E \cap B_i$. Our first observation is that for N sufficiently large, most of an atom A in $\bigvee_N^{N'} \phi^k\alpha$ intersects each B_i in an $\mathscr{E}$-tubular subset. For this it suffices to prove:

LEMMA 2.1. *Suppose that α is a partition satisfying* (RA), *B is a parallelogram and $\delta > 0$ is given. There exists an N_1 such that for any $N' > N \geqq N_1$ and δ-almost every atom $A \in \bigvee_N^{N'} \phi^k\alpha$ there is a subset $E \subset A$ with*

$$\mu(E)/\mu(A) > 1 - \delta, \tag{1}$$

$$\text{E intersects B in an $\mathscr{E}$-tubular subset.} \tag{2}$$

PROOF. By the invariance of the foliations it follows that operating by ϕ does not affect the tubularity of intersections. Let G_k denote the *non*-tubular intersections of atoms of $\phi^k\alpha$ with B; that is, G_k is the union over $\phi^kA \in \phi^k\alpha$, of the non-tubular intersections of B with ϕ^kA. The non-tubular intersections consist of all $x \in B \cap \phi^kA$ for which $\mathscr{E}_x \cap B$ is not contained entirely in $B \cap \phi^kA$. To estimate the size of G_k it is convenient to apply ϕ^{-k}. Since B is a parallelogram and the foliation $\mathscr{E}$ contracts exponentially with negative powers of ϕ, any point in a non-tubular intersection of A with $\phi^{-k}B$ must be within distance d_k of the boundary of A, where d_k satisfies

$$d_k \leqq C\cdot|\lambda_2|^{-k}. \tag{3}$$

From (3) it follows that

(4) $$\mu(G_k) \leqq C \cdot |\lambda_2|^{-k}$$

since the boundary of A is smooth. Now let N_1 be chosen large enough so that

(5) $$\mu(G) \leqq \sum_{k=N_1}^{\infty} \mu(G_k) \leqq \delta^2$$

where $G = \cup_{N_1}^{\infty} G_k$. Now for $N' \geqq N \geqq N_1$, δ-almost every atom of $\bigvee_N^{N'} \phi^k \alpha$ intersects G in a set of relative measure at most δ, since otherwise, adding up over those atoms that intersect G in a set of relative measure greater than δ, we would contradict (5). This clearly implies the existence of $E \subset A$ satisfying (1) and (2) for δ-almost every atom of $\bigvee_N^{N'} \phi^k \alpha$. □

LEMMA 2.2. *Let $\delta > 0$, α be a partition satisfying RA and B be a parallelogram. Then there is an N_2 such that for any $N' > N \geqq N_2$, and δ-almost every $A \in \bigvee_N^{N'} \phi^k \alpha$ we have*

(6) $$|\mu(A \cap B)/\mu(A) - \mu(B)| \leqq \delta.$$

PROOF. This follows immediately from the fact that ϕ is a K-automorphism, a result due to V. A. Rohlin [13]. It can also be obtained, in this case, by using the irrationality of the eigenvalues and Kronecker's theorem on uniform distribution. □

These first two lemmas enable us to reduce the problem of mapping atoms of $\bigvee_N^{N'} \phi^k \alpha$ onto X in a way which keeps positive orbits of nearby points close together, to a local problem of mapping $\mathscr{E}$ tubular subsets of parallelograms in such a fashion. It is the generalization of this next lemma that causes the difficulties in the generalizations and so, in spite of its simplicity in this setting, we call it:

MAIN LEMMA. *Given $\delta_1 > 0$ there is a $\delta_2 > 0$ such that if B is a parallelogram of diameter less than δ_2 and E is an $\mathscr{E}$-tubular subset of B, there is a one-to-one, onto mapping $\theta: E \to B$ such that*

(7) *θ is measure preserving*

(8) $$d(\phi^k \theta x, \phi^k x) < \delta_1, \text{ for all } k \geqq 0, x \in E.$$

PROOF. If δ_2 is small enough then (8) will be satisfied provided that $\theta x \in \mathscr{C}_x \cap B$, since distances contract as ϕ operates on $\mathscr{C}$. Let x_0 be some fixed point in the interior of B, and define θ on $E \cap \mathscr{C}_{x_0} \cap B$ as some one-to-one mapping of $E \cap \mathscr{C}_{x_0} \cap B$ onto $\mathscr{C}_{x_0} \cap B$ which preserves linear Lebesgue measure. There is a natural one to one mapping $\pi_{x_0,x}$ between $\mathscr{C}_{x_0} \cap B$ and $\mathscr{C}_x \cap B$ for any $x \in B$, which satisfies $\pi_{x_0,x}(y) \in (\mathscr{E}_y \cap B) \cap (\mathscr{C}_x \cap B)$ and which preserves the linear Lebesgue measures. Use these $\pi_{x_0,x}$ to define θ on $\mathscr{C}_x \cap B$ by $\pi_{x_0,x} \theta \pi_{x_0,x}^{-1}$. This is possible since E is a tubular set. By Fubini's theorem, the mapping θ so defined on $E \cap B$ is measure preserving and so (7) holds. As we have remarked, (8) holds if δ_2 is small enough, where this choice clearly depends only on δ_1 and the eigenvalue λ_1. □

LEMMA 2.3. *Let $\varepsilon, \varepsilon' > 0$ and let α be a partition that satisfies* (RA). *Then there is an N such that for all $N' \geqq N$ and for ε-almost every atom $A \in \bigvee_N^{N'} \phi^k \alpha$ there is a set $E \subset A$, and a one-to-one mapping θ of E onto X such that*

$$\mu(E)/\mu(A) > 1 - \varepsilon, \tag{9}$$

$$\theta \text{ is } \varepsilon\text{-measure preserving}, \tag{10}$$

$$\text{for any } k \geqq 0,\ x \in E,\ \mathrm{d}(\phi^k x, \phi^k \theta x) \leqq \varepsilon'. \tag{11}$$

PROOF. Apply the main lemma with δ_1 replaced by ε' to obtain δ_2, and let $\beta = \{B_0, B_1, \cdots B_b\}$ be a partition of X such that the B_i are parallelograms, for $1 \leqq i \leqq b$, with diameter no greater than δ_2, while $\mu(B_0) < \varepsilon/10$. Such partitions clearly exist. Combining Lemma 2.1 and 2.2 we can find an N such that for ε-a.e. atom A of $\bigvee_N^{N'} \phi^k \alpha$ there is an $E \subset A$ with (9) holding, and for all $B_j \in \beta$, $1 \leqq j \leqq b$,

$$\sum_{j=1}^{b} \left| \mu(E \cap B_j)/\mu(E) - \mu(B_j) \right| < \varepsilon, \tag{12}$$

and furthermore, E intersects B_j $\mathscr{E}$ tubularly, and $E \cap B_0 = \varnothing$. Apply the main lemma to $E \cap B_j$, for each j separately, $1 \leqq j \leqq b$ and combine the resulting mappings to a mapping $\theta: E \to X$.

The main lemma implies that (11) holds while (10) follows from (12) and the fact that θ is measure preserving on each $E \cap B_j$. □

For further generalizations note that in the proof of this lemma it was not essential that the partial mappings, given by the main lemma, were actually measure preserving. It would have been sufficient to know that they were $\varepsilon/10$-measure-preserving, for example, if we had, in addition, replaced ε by $\varepsilon/10$ in (12). The next, final lemma will bring us directly to the proof of the theorem.

LEMMA 2.4. *Given a partition α that satisfies (RA) and an $\varepsilon > 0$, there is an N such that for all $N' \geqq N$ and ε-a.e. atom A of $\bigvee_N^{N'} \phi^k \alpha$*

$$\mathrm{d}(\{\phi^{-i}\alpha \mid A\}_1^n, \quad \{\phi^{-i}\alpha\}_1^n) \leqq \varepsilon \tag{13}$$

for all $n \geqq 1$. In other words, any partition that satisfies (RA) is VWB.

PROOF. We would like to use Lemma 2.3 in order to apply Lemma 1.3. First, observe that the $\{\phi^{-i}\alpha\}$-names, as well as the $\{\phi^{-i}\alpha|E_i\}$-names of a point x are determined by the α-name of $\phi^i x$. Since the atoms of α have smooth boundaries if ε' is small enough, conclusion (11) of Lemma 2.3 implies hypothesis (17). This is a consequence of the fact that if ε' is small enough, the set of points x, y such that $d(x,y) < \varepsilon'$ and x and y lie in different atoms of α is of arbitrarily small measure, and averaging over $1 \leqq j \leqq n$ leads to (17) in Lemma 1.3. Now choose N large enough so that Lemma 2.3 holds with ε replaced by $\varepsilon/16$ and ε' small enough to ensure (17) in Lemma 1.3 with $\varepsilon/16$. Extending θ arbitrarily to $A \setminus E$ and applying Lemma 1.3, we get (13). □

PROOF OF THEOREM 2.1. Lemma 4 and Theorem A say that if α is any partition that satisfies (RA), ϕ is a B-shift on $(X, \bigvee_{-\infty}^{\infty}\phi^k\alpha, \mu)$ and since one can easily construct an increasing sequence $\alpha_1 \subseteq \alpha_2 \subseteq \cdots$ of such partitions with $\bigvee_1^{\infty}\alpha_i$ the full σ-algebra of Borel sets, Theorem B completes the proof. □

3. Geodesic flows

A classical example which has played a very important role in the development of ergodic theory is the geodesic flow on a surface of negative curvature. Let M be a complete Riemannian manifold and X its unit tangent bundle; that is, $x \in X$ is a pair (p,u) where p is a point of M and u is a unit tangent vector to M at p. The geodesic flow $\phi : X \to X$ is defined by $\phi(p,u) = (p',u')$ where to obtain (p',j') one constructs the geodesic on M that passes through p in the direction u, and then one moves out along the geodesic for a distance t, measured in arc length to reach the point p'; u' is the direction of the tangent to the geodesic so constructed at p'. The completeness of M guarantees that ϕ_t is well-defined for all t. Clearly $\phi_t\phi_s = \phi_{t+s}$, and one easily checks that ϕ preserves the natural volume on X, denoted by μ, so that we have a measure-preserving flow. Substantial progress was made in the study of the ergodic properties of such flows in the 1930's by G. Hedlund, E. Hopf and others (cf. [3], [4] and [5]). After a period of quiescence, the subject was revived and advanced about a decade ago with the work of Anosov, Arnold and Sinai, a detailed account of their work may be found in [2] which will be the basic reference for this section. Our goal in this section is to prove that if M is compact, two-dimensional with negative curvature, then ϕ is a Bernoulli flow; that is, ϕ_t is a B-shift for each t. To minimize the geometrical argument at this stage, we do not develop the most general result possible. We shall follow closely the line of proof in §2 and begin by fixing the notation and generalizing the notion of a parallelogram.

Let M be a compact two dimensional Riemannian manifold of class C^3 with curvature K that satisfies

$$K < 0 \tag{1}$$

throughout M. As above, X will denote the unit tangent bundle of M and $\phi_t: X \to X$ the geodesic flow. If dA is the areal measure on M then ϕ_t preserves $dAdu$ where du is the usual Lebesgue measure on the circle. Denote $dAdu$ by $d\mu$.

The key to the mixing properties of ϕ_t lies in the existence of expanding and contracting foliations invariant under ϕ_t. These were used more or less explicitly by Hedlund and Hopf to prove ergodicity, and then were the main tool in the proof of Anosov and Sinai that ϕ_t is a K-flow. The basic properties of these foliations were known to J. Hadamard and E. Cartan, although not explicitly as foliations; for a brief discussion with references see [2, §2, Example C]. For the properties we shall need of these foliations see [2, §22–23]. There are two foliations of X into differentiable curves, denoted by $\mathscr{E}$ and $\mathscr{C}$, which are mutually transversal and transversal to the flow direction. They are invariant under ϕ_t; i.e., $\phi_t\mathscr{E}_x = \mathscr{E}_{\phi_t x}$, $\phi\mathscr{C}_x = \mathscr{C}_{\phi_t x}$ and ϕ_t expands distances along $\mathscr{E}$ exponentially while it contracts distances along $\mathscr{C}$ exponentially as $t \to +\infty$. Naturally, by its very definition, ϕ_t preserves another foliation, namely the foliation of X by the orbits of $\{\phi_t\}$, and ϕ_t preserves distances along this foliation. The parallelograms that played a central role in the main lemma of §2 are replaced by *P-sets* which we now define. Start with a small connected segment $\gamma \subset \mathscr{C}_{x_0}$ of the $\mathscr{C}$-foliation, form the surface $\bigcup_{t=0}^{t_0}\phi_t\gamma = C$ and then move $\mathscr{C}$ along $\mathscr{E}$ to another surface of that type. That is, for each $x \in C$, take a segment of $\mathscr{E}_x$ that starts at x and ends at $e(x)$ in such a way that

$$\bigcup_{x \in C} e(x) \subset \bigcup_{t=0}^{t_1} \phi_t\gamma', \qquad \gamma' \subset \mathscr{C}_{x'} \tag{2}$$

for some x'. A P-set is then a set of the form

$$B = \bigcup_{x \in C} \mathscr{E}_{x,e(x)} \tag{3}$$

where $\mathscr{E}_{x_1,x_2}$ is the segment of $\mathscr{E}_{x_1}$ that connects x_1 to $x_2 \in \mathscr{E}_{x_1}$. We assume that the lengths of $\mathscr{E}_{x,e(x)}$ are bounded away from zero and infinity.

First we establish the analogues of Lemmas 2.1 and 2.2 in order to reduce our problem to a local one. The convention that $\mathscr{E}_x \cap A$ denotes the connected component of $\mathscr{E}_x \cap A$ that contains x will be made as in §2. An *$\mathscr{E}$-tubular subset* of a P-set will be defined as before; namely, E is called an $\mathscr{E}$-tubular subset of a P-set B if $E \subset B$, and $x \in E$ implies $\mathscr{E}_x \cap B \subset E$. The partitions which we shall show are VWB should satisfy some mild regularity assumptions such as: (RA) α is a finite partition each atom of which is connected with dense interior and has a smooth boundary.

For the remainder of this section we fix a time t_0, denote $\phi_{t_0} = \phi$ and concentrate on showing that ϕ is a B-shift by following the pattern of proof in §2.

LEMMA 3.1. *Suppose that α is a partition that satisfies (RA), B is a P-set and $\delta > 0$ is given. Then there is an N_1 such that for any $N' \geqq N \geqq N_1$, for δ-a.e. atom A of $\bigvee_N^{N'} \phi^k \alpha$ there is a subset $E \subset A$ with*

(1) $$\mu(E)/\mu(A) > 1 - \delta,$$

(2) *$E \cap B$ is an $\mathscr{E}$-tubular subset of B.*

PROOF. For each k, consider $\phi^k\alpha$ and let $G_k \subset B$ denote that part of B that is intersected by an atom $A^{(k)}$ of $\phi^k\alpha$ in a non-$\mathscr{E}$-tubular fashion; that is, $x \in G_k$ if for some atom $A^{(k)} \in \phi^k\alpha$

(3) $$x \in A^{(k)} \cap B, \text{ and there exists } y \in \mathscr{E}_x \cap B,\ y \notin A^{(k)}.$$

We wish to estimate the size of G_k, and to this end note that since the foliations are ϕ-invariant, the tubularity of sets is not affected by applying the transformation ϕ. From the definition (3) we see that $\phi^{-k}G_k$ lies in the set of points whose distance to the boundary of some atom of α is no greater than d_k, where d_k is the maximal length of a connected segment of an $\mathscr{E}$-fiber in $\phi^{-k}B$. Since α satisfies (RA) the measure of this set is thus bounded by a constant times d_k. The exponential contraction of $\mathscr{E}$ as $t \to -\infty$, and the definition of a P-set imply that

(4) $$\sum_N^\infty d_k \to 0 \text{ as } N \to \infty.$$

Since ϕ is measure-preserving we can conclude that $\sum_N^\infty \mu(G_k)$ tends to zero as N tends to infinity. Choosing N_1 so that

(5) $$\sum_{N_1}^\infty \mu(G_k) < \delta^2$$

we have that δ-a.e. atom A of $\bigvee_N^{N'} \phi^k\alpha$ $(N \geqq N_1)$ intersects $\bigcup_{N_1}^\infty G_k$ in a set of measure at most $\delta\mu(A)$; letting $E = A \cap (X \setminus \bigcup_{N_1}^{\infty} G_k)$ for the good atoms, we have that (1) and (2) hold. □

LEMMA 3.2. *Let α be a partition that satisfies (RA), B a P-set and suppose that $\delta > 0$ is given. There is an N_2 such that for any $N' \geqq N \geqq N_2$ and δ-a.e. atom A of $\bigvee_N^{N'} \phi^k \alpha$ we have*

$$(6) \qquad |\mu(B \cap A/\mu(A) - \mu(B)| \leqq \delta.$$

PROOF. D. V. Anosov and Ya. G. Sinai have proved that ϕ is a K-automorphism [2, §22, and Appendix] whence (6) follows. □

At one stroke we have been able to reduce the problem of showing that a partition α that satisfies (RA) if VWB to a local problem involving the local structure of $\mathscr{E}$ and $\mathscr{C}$. One can extract the geometric information that is needed to prove the main lemma from the work of Anosov and Sinai on "absolute continuity" of the foliations $\mathscr{E}$ and $\mathscr{C}$, and indeed this approach is necessary for higher dimensional generalizations. However, for dimension 2, we can give a cleaner argument by using the more precise results of E. Hopf. Translating the results of [5, §3, 7] into our terminology leads to the following:

THEOREM H. *Local C^1 coordinates can be introduced in X, (α_1, α_2, s) such that*

$$(7) \qquad \phi(\alpha_1, \alpha_2, s) = (\alpha_1, \alpha_2, s + t)$$

and the surfaces α_1 = constant, α_2 = constant correspond to the surfaces obtained by taking a segment γ of $\mathscr{E}$ or $\mathscr{C}$, respectively, and forming $\bigcup_{|t| < t_0} \phi_t \gamma$.

The content of the theorem lies of course in the statement that the coordinates are C^1. We are now in a position to prove

MAIN LEMMA. *Given $\varepsilon_1, \delta_1 > 0$, there is a $\delta_2 > 0$ such that if B is a P-set of diameter less than δ_2 and E is an $\mathscr{E}$-tubular subset of B, there is a one-to-one mapping $\theta: E \to B$ such that*

$$(8) \qquad \theta \text{ is } \varepsilon_1\text{-measure-preserving},$$

$$(9) \qquad d(\phi^k \theta x, \phi^k x) < \delta_1 \text{ for all } k \geqq 0,\ x \in E.$$

PROOF. In the coordinates of Theorem H, the invariant measure $d\mu$ takes the form

$$(10) \qquad d\mu = p(\alpha_1, \alpha_2) d\alpha_1 \, d\alpha_2 \, dt$$

for some smooth function p. First, choose δ_2 small enough so that $p(\alpha_1, \alpha_2)$ is nearly constant on sets of diameter less than δ_2, where nearly constant is relative to ε_1 in the sense that we require for (α_1, α_2), (α_1, α_2) in the same set B:

$$|p(\alpha_1,\alpha_2)/p(\alpha_1,\alpha_2) - 1| \leqq \varepsilon_1/10. \tag{11}$$

Then, choose δ_2 still smaller, if necessary, to ensure that if $d(x,y) < \delta_2$ and $\alpha_2(x) = \alpha_2(y)$, then for all $k \geqq 0$

$$d(\phi^k x, \phi^k y) \leqq \delta_1. \tag{12}$$

The reason that this is possible is that if $\alpha_2(x) = \alpha_2(y)$, then for some t_1 bounded by a constant times $d(x,y)$, $x = (\alpha_1(x), \alpha_2(x), s(x))$ and $y' = (\alpha_1(y). \alpha_2(y), s(u) + t_1)$ lie on the same contracting fiber of $\mathscr{C}$; thus (12) certainly holds with y replaced by y'. But if δ_2 is small enough, taking note of (7) we would then get (12) as written.

Having chosen δ_2, we construct θ in a fashion similar to that used in the proof of the main lemma in §2. Let us call an $\mathscr{E}$-tubular subset $E \subset B$ *simple* if it has constant width in the direction t. For a *simple* $\mathscr{E}$-tubular subset $E \subset B$, the planes α_2 = constant intersect E in sections of equal planar measure, and since B itself is a simple $\mathscr{E}$-tubular set, we can map E onto B along planes α_2 = constant in a fashion that preserves the measure $d\alpha_1\, d\alpha_2\, dt$. Now approximate an arbitrary $\mathscr{E}$-tubular subset by simple ones, use (11) to go from $d\alpha_1\, d\alpha_2\, dt$ to $d\mu$ and note that (12) implies (9) if x and θx lie on the same surface, α_2 = constant to complete the proof. □

LEMMA 3.3. *Let $\varepsilon, \varepsilon' > 0$ and α, a partition satisfying (RA), be given. Then there is an N such that for all $N' \geqq N$ and ε-a.e. atom $A \in \bigvee_N^{N'} \phi^i \alpha$ there is a set $E \subset A$ and a one-to-one mapping $\theta: A \to X$ such that*

$$\mu(E)/\mu(A) > 1 - \varepsilon \tag{13}$$

$$\theta \text{ is } \varepsilon\text{-measure-preserving} \tag{14}$$

$$\text{for any } k \geqq 0,\ x \in E,\ d(\phi^k \theta x, \phi^k x) \leqq \varepsilon'. \tag{15}$$

PROOF. A standard covering argument shows that for any δ, we can find a finite number of disjoint P-sets, $B_1, B_2, \cdots B_b$, of diameter less than δ such that $B_0 = X \setminus \bigcup_1^b B_i$ has measure less than δ. Using the partition $\beta = \{B_0, B_1, \cdots B_b\}$ and the preceding lemmas, the proof of Lemma 2.3 can be applied to yield Lemma 3.3. □

LEMMA 3.4. *Any partition α that satisfies (RA) is VWB for ϕ.*

PROOF. The deduction of Lemma 3.4 from Lemma 3.3 is the same as the proof of Lemma 2.4 □

THEOREM 3. *The geodesic flow on a compact C^2-manifold of negative curvature is isomorphic to a Bernoulli flow.*

PROOF. One can find a sequence of partitions $\alpha_1 \subset \alpha_2 \subseteq \cdots \subseteq \alpha_i \subseteq \cdots$ that satisfy (RA) and such that $\bigvee_1^\infty \alpha_n$ is the full σ-algebra of Borel sets. Then Lemma 3.4, and Theorems A and B imply that $\phi = \phi_{t_0}$ is a Bernoulli shift. Therefore, ϕ_t is isomorphic to a Bernoulli flow. □

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